Continous Time Finance I
Solution to 1st Minitest

by Yihui Ni

Group A Exercise 1

How can one characterize the Wiener process (explain in two ways)?

1. The Wiener process has $\overbrace{\textbf{independent and stationary increments (or Levy process)}}^{\text{0.5 Points}}$, and the increments are $\underbrace{\textbf{normally distributed}}_{\text{0.5 Points}}$.
2. The Wiener process has **independent and stationary increments**, and all $\underbrace{\textbf{paths are continuous}}_{\text{1 Points}}$.

Group A Exercise 2

Let (W_t) be a Wiener process. When is $at + 2W_t^2$ martingale (with proof)?

Proof 1:

$$\begin{aligned}
E(at + 2W_t^2|\mathcal{F}_s) = E(at + 2(W_t - W_s + W_s)^2|\mathcal{F}_s) \\
&= at + 2E((W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2|\mathcal{F}_s) \\
&= at + 2[E((W_t - W_s)^2|\mathcal{F}_s) + E(2(W_t - W_s)W_s|\mathcal{F}_s)) + E(W_s^2|\mathcal{F}_s)] \\
&= at + 2[E((W_t - W_s)^2) + 2W_sE(W_t - W_s) + W_s^2] \\
&= at + 2E((W_t - W_s)^2) + 4W_s \cdot 0 + 2W_s^2 \\
&= at + 2(t - s) + 2W_s^2 \\
&= at + 2t - 2s + 2W_s^2
\end{aligned}$$

For $E(at + 2W_t^2|\mathcal{F}_s) = as + 2W_s^2$, it must hold $a = -2$. □

Proof 2:

Show that $E(W_t^2 - t|\mathcal{F}_s)$ is a martingale, then $E(2W_t^2 - 2t|\mathcal{F}_s)$ is also a martingale.

$$\begin{aligned}
E(W_t^2 - t|\mathcal{F}_s) &= E((W_t - W_s + W_s)^2 - t|\mathcal{F}_s) \\
&= E((W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2 - t|\mathcal{F}_s) \\
&= E((W_t - W_s)^2|\mathcal{F}_s) + E((W_t - W_s)W_s|\mathcal{F}_s)) + E(W_s^2|\mathcal{F}_s)] - t \\
&= E((W_t - W_s)^2) + W_s E(W_t - W_s) + W_s^2 - t \\
&= (t - s) + 4W_s \cdot 0 + 2W_s^2 - t \\
&= W_s^2 - s
\end{aligned}$$

Therefore, $E(2W_t^2 + at|\mathcal{F}_s)$ is a martingale if $E(2W_t^2 + at|\mathcal{F}_s) = E(2W_t^2 - 2t|\mathcal{F}_s)$, so $a = -2$. $\square$

Group B Exercise 1

Why are mean and variance of Levy processes linear functions of t?

For liniarity, additivity property need to be fullfilled:

$$X_{s+t} - X_0 = (X_{s+t} - X_s) + (X_s - X_0)$$

mean (1 Points)

$$E(X_{s+t} - X_0) = \underbrace{E(X_{s+t} - X_s)}_{\textbf{stationarity}\Rightarrow E(X_t)} + \underbrace{E(X_s - X_0)}_{E(X_s)}$$

$$E(X_{s+t} - X_0) = E(X_t) + E(X_s)$$

variance (1 Points)

For the variance it is important to mention the independent increment.

$$V(X_{s+t} - X_0) \overset{\substack{\textbf{independent} \\ \textbf{increments (0.5 Points)}}}{=} \underbrace{V(X_{s+t} - X_s)}_{\textbf{stationarity}\Rightarrow V(X_t)} + \underbrace{V(X_s - X_0)}_{V(X_s)}$$

$$V(X_{s+t} - X_0) \overset{\substack{\textbf{independent} \\ \textbf{increments}}}{=} V(X_t) + V(X_s)$$

Group B Exercise 2

Let (W_t) be a Wiener process. When is $e^{\alpha W_t + \beta t}$ a martingale ? (with proof)

Proof:

$$\begin{aligned}
E(e^{\alpha W_t+\beta t}|\mathcal{F}_s) &= e^{\beta t}E(e^{\alpha W_t}|\mathcal{F}_s) \\
&= e^{\beta t}E(e^{\alpha W_s+\alpha(W_t-W_s)}|\mathcal{F}_s) \\
&= e^{\beta t}e^{\alpha W_s}E(e^{\alpha(W_t-W_s)}) \\
&= e^{\beta t}e^{\alpha W_s}e^{\alpha^2(t-s)/2} \\
&= e^{\beta t+\alpha W_s+\alpha^2 t/2-\alpha^2 s/2} \\
&= e^{\alpha W_s+\underbrace{(\beta t+\alpha^2 t/2-\alpha^2 s/2)}_{\beta s}}
\end{aligned}$$

For $E(e^{\alpha W_t+\beta t}|\mathcal{F}_s) = e^{\alpha W_s+\beta s}$, it must hold $\beta = -\frac{\alpha^2}{2}$ $\square$

Group C Exercise 1

How fast do the paths of Wiener process increase on average?

Proof:

Find how fast X_t is moving if $t \to \infty$. For this we use the property $\frac{X_t}{t} \xrightarrow{P} E(X_1)$, where $E(X_1) = 0$.
By using Chebyshev's inequality* :

$$\begin{aligned}
P(|\frac{X_t}{t}-0| > \epsilon) &\leq \frac{1}{\epsilon^2}V(\frac{X_t}{t}) \\
&= \frac{1}{\epsilon^2 t^2}V(X_t) \\
{**} &= \frac{1}{\epsilon^2 t^2}(\underbrace{E(X_t^2)}_{=t}-\underbrace{E(X_t)^2}_{=0}) \\
&= \frac{1}{\epsilon^2 t^2} \\
&= \frac{1}{\epsilon^2 t}
\end{aligned}$$

$\square$

For $t \to \infty$ and $\epsilon > 0$, a finite positive number, $\frac{1}{\epsilon^2 t} \to 0$. This implies that $P(|\frac{X_t}{t} - E(X_1)| > \epsilon) \leq 0$, so $P(|\frac{X_t}{t} - E(X_1)| > \epsilon) \to 0$. This is only the case if $(\frac{X_t}{t} - 0) \to 0$ or $\frac{X_t}{t} \to 0$, indicating that the path X_t does not increase faster than linear t.

Group C Exercise 2

Let (W_t) be a Wiener process and $s < t$. Find $E(e^{3W_t - W_s}|\mathcal{F}_s)$

$$\begin{aligned} E(e^{3W_t - W_s}|\mathcal{F}_s)) &= E(e^{3(W_t - W_s)+2W_s}|\mathcal{F}_s)) \\ &= e^{2W_s}E(e^{3(W_t-W_s)}) \\ &= e^{2W_s}e^{9(t-s)/2} \\ &= e^{2W_s+9(t-s)/2} \end{aligned}$$

Group A, B, C Exercise 3

Show that the Wiener process has finite quadratic variation

Proof:
Show for $0 = t_0 < t_1 < \ldots < t_k = t$

$$\sum_i (W_{t_i} - W_{t_{i-1}})^2 \xrightarrow{\text{P}} t$$

By using Chebyshev's inequality show $\text{P}(|\sum_i (W_{t_i} - W_{t_{i-1}})^2 - t| > \epsilon) \xrightarrow{!} 0$

$P(|\sum_i (W_{t_i} - W_{t_{i-1}})^2 - t| > \epsilon) \leq \frac{V(\sum_i (W_{t_i} - W_{t_{i-1}})^2)}{\epsilon^2}$

Version 1:

$$\begin{aligned} V(\sum_i (W_{t_i} - W_{t_{i-1}})^2) &\overset{\substack{\textbf{independent} \\ \textbf{increments (0.5 Points)}}}{=} \sum_i (V(W_{t_i} - W_{t_{i-1}})^2) \\ {\color{red}**} &= \sum_i (E((W_{t_i} - W_{t_{i-1}})^4) - E((W_{t_i} - W_{t_{i-1}})^2)^2) \\ {\color{red}***} &\leq \sum_i 3\underbrace{(t_i - t_{i-1})}_{\sigma^2}{}^2 \\ &= \sum_i 3(t_i - t_{i-1})(t_i - t_{i-1}) \\ &\leq \underbrace{\max(t_i - t_{i-1})}_{\to 0 \text{ intervals gets smaller}} \underbrace{\sum_i (t_i - t_{i-1})}_{=t} \\ &= 0 \end{aligned}$$

□

Version 2:

$$
\begin{aligned}
V(\sum_i (W_{t_i} - W_{t_{i-1}})^2) &\overset{\substack{\textbf{independent}\\ \textbf{increments (0.5 Points)}}}{=} \sum_i (V(W_{t_i} - W_{t_{i-1}})^2) \\
&** = \sum_i (E((W_{t_i} - W_{t_{i-1}})^4) - E((W_{t_i} - W_{t_{i-1}})^2)^2) \\
&= \sum_i 3(t_i - t_{i-1})^2 - (t_i - t_{i-1})^2 \\
&= \sum_i 2(t_i - t_{i-1})^2 \\
&= \sum_i 2(t_i - t_{i-1})(t_i - t_{i-1}) \\
&\leq \underbrace{\max(t_i - t_{i-1})}_{\to 0 \text{ intervals gets smaller}} \underbrace{\sum_i (t_i - t_{i-1})}_{=t} \\
&= 0
\end{aligned}
$$

$\square$

* Chebyshev's inequality: $P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

** $V(X) = E(X^2) - E(X)^2$
$V(X^2) = E(X^4) - E(X^2)^2$

*** $E((\sigma X)^4) = \sigma^4 E(X^4) = 3\sigma^4 = 3 \cdot V(X)^2$
$E((W_{t_i} - W_{t_{i-1}}))^4 = 3 \cdot V(W_{t_i} - W_{t_{i-1}})^2$

Continous Time Finance I
Solution to 2nd Minitest

by Yihui Ni

Group A Exercise 1

Find $\int_0^t (s - s^2)\, d\sqrt{s}$ *?*

$$\begin{aligned}
\int_0^t (s - s^2)\, d\sqrt(s) &= \int_0^t (s - s^2) \frac{1}{2} s^{-\frac{1}{2}}\, ds \\
&= \frac{1}{2}[\int_0^t s^{\frac{1}{2}}\, ds - \int_0^t s^{\frac{3}{2}}\, ds] \\
&= \frac{1}{2}[\, s^{\frac{1}{2}} \,|_0^t - s^{\frac{3}{2}} \,|_0^t \,] \\
&= \frac{1}{2}[\, \frac{2}{3} s^{\frac{3}{2}} \,|_0^t - \frac{2}{5} s^{\frac{5}{2}} \,|_0^t \,] \\
&= \frac{1}{3} t^{\frac{3}{2}} - \frac{1}{5} t^{\frac{5}{2}}
\end{aligned}$$

Group A Exercise 2

Let $X_t = \int_0^t f(t)\, dW_t$ *(*f *being a step function). Show that* $V(X_t) = \int_0^t f^2(s)\, ds$*.*

Proof 1:
$f(s) = a\mathbf{1}_{(0,t_1]} + b\mathbf{1}_{(t_1,t]}$
$\int_0^t f(s)\, dWs = aW_{t_1} + b(W_t - W_{t_1})$

$$V(\int_0^t f(s)\,dWs) = E(a^2W_{t_1}^2 + b^2(W_t - W_{t_1})^2 + 2abW_{t_1}(W_t - W_{t_1}))$$
$$= E(a^2W_{t_1}^2) + E(b^2(W_t - W_{t_1})^2) + E(2abW_{t_1}(W_t - W_{t_1}))$$
$$= E(a^2W_{t_1}^2) + E(b^2(W_t - W_{t_1})^2) + 2ab\,\underbrace{E((W_{t_1})(W_t - W_{t_1}))}_{=0 \text{ reason see } *}$$
$$= a^2t_1 + b^2(t - t_1)$$
$$= \int_0^t f^2(s)\,ds$$

* $E(W_{t_1}(W_t - W_{t_1})) = E(E(W_{t_1}(W_t - W_{t_1})|\mathcal{F}_{t_1})) = E(W_{t_1}E((W_t - W_{t_1})|\mathcal{F}_{t_1})) = 0$
... Martingale Property
Or one could also argue that $W_{t_1} = (W_{t_1} - W_0)$ and $(W_t - W_{t_1})$ are independent. □

Proof 2:
$X_t = \int_0^t f(s)\,dWs = \sum_{i=1}^n a_i(W_{t_i} - W_{t_{i-1}})$ ... Wiener Integral

$$V(\int_0^t f(s)\,dWs) = V(\sum_{i=1}^n a_i(W_{t_i} - W_{t_{i-1}}))$$
$$\overset{\substack{\textbf{independent}\\\textbf{increments}}}{=} \sum_{i=1}^n a_i^2 V(W_{t_i} - W_{t_{i-1}})$$
$$= \sum_{i=1}^n a_i^2(t_i - t_{i-1})$$
$$= \int_0^t f^2(s)\,ds$$

□

Group B Exercise 1

Find $\int_0^t s^2e^s\,ds^3$ *? (Solutions 1)*

Unfortunately, there has been a typo in this exercise (correct version see below). Accordingly, we will be indulgent while correcting.

Find $\int_0^t s^{-2}e^s\,ds^3$ *? (Solution 2)*

Solution 1
Integration by parts: $\int_0^t f'(x)g(x)\,ds = f(x)g(x)|_0^t - \int_0^t f(x)g(x)'\,ds$

$$
\begin{aligned}
\int_0^t s^2 e^s \, ds^3 &= \int_0^t s^2 e^s 3s^2 \, ds \\
&= 3\int_0^t e^s s^4 \, ds \\
&= 3[e^s s^4 \,|_0^t - \int_0^t e^s 4s^3 \, ds] \\
&= 3[e^s s^4 \,|_0^t - 4[e^s s^3 \,|_0^t - \int_0^t e^s 3s^2 \, ds]] \\
&= 3[e^s s^4 \,|_0^t - 4[e^s s^3 \,|_0^t - 3[e^s s^2 \,|_0^t - \int_0^t e^s 2s \, ds]]] \\
&= 3[e^s s^4 \,|_0^t - 4[e^s s^3 \,|_0^t - 3[e^s s^2 \,|_0^t - 2[e^s s \,|_0^t - \int_0^t e^s \, ds]]]] \\
&= 3[e^s s^4 \,|_0^t - 4[e^s s^3 \,|_0^t - 3[e^s s^2 \,|_0^t - 2[e^s s \,|_0^t - e^s \,|_0^t]]]] \\
&= 3[e^t t^4 - 4[e^t t^3 - 3[e^t t^2 - 2[e^t t - [e^t - 1]]]]] \\
&= 3[e^t t^4 - 4e^t t^3 + 12 e^t t^2 - 24 e^t t + 24 e^t - 24]
\end{aligned}
$$

Solution 2

$$
\begin{aligned}
\int_0^t s^{-2} e^s \, ds^3 &= \int_0^t s^{-2} e^s 3s^2 \, ds \\
&= 3\int_0^t e^s \, ds \\
&= 3(e^s \,|_0^t) \\
&= 3(e^t - 1)
\end{aligned}
$$

Group B Exercise 2

Show that the Wiener Integral (of a step function) is a normally distributed random variable with expectation zero.

$f(s) = \sum_{i=1}^n a_i \mathbf{1}_{(t_{i-1}, t_i]}$

$X_t = \int_0^t f(s) \, dWs = \sum_{i=1}^n a_i (W_{t_i} - W_{t_{i-1}})$ ... Wiener Integral

Wiener Integral is a linear combination of normally distributed independent random variables, therefore it is also a normally distributed random variable.

$$
\begin{aligned}
E(X_t) &= E(\sum_{i=1}^{n} a_i(W_{t_i} - W_{t_{i-1}})) \\
&= \sum_{i=1}^{n} a_i \underbrace{E(W_{t_i} - W_{t_{i-1}})}_{\text{=0 reason see 2)}} \\
&= 0
\end{aligned}
$$

2) ($W_{t_i} - W_{t_{i-1}}$) is normally distributed and independent

Group C Exercise 1

Find $\int_0^t (e^{-s} - 1)\, de^s$ *?*

$$
\begin{aligned}
\int_0^t (e^{-s} - 1)\, de^s &= \int_0^t (e^{-s} - 1)e^s\, ds \\
&= \int_0^t 1 - e^s\, ds \\
&= \int_0^t 1\, ds - \int_0^t e^s\, ds \\
&= s|_o^t - e^s|_o^t \\
&= t - e^t + 1
\end{aligned}
$$

Group C Exercise 2

Show that the process $X_t = \int_0^t f(s)\, dW_s$ *(f being a step function) has indepentent increments.*

Proof 1: We need to show, for u < t, that $X_t - X_u$ is independent of $\mathcal{F}_u$ (the past).

$$
\begin{aligned}
E(X_t - X_u|\mathcal{F}_u) &= E(\int_0^t f(s)\,dW_s - \int_0^u f(s)\,dW_s|\mathcal{F}_u) \\
&= E(\int_u^t f(s)\,dW_s|\mathcal{F}_u) \\
&= E(\sum_{i=s+1}^{t} a_i(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_u) \\
&= \sum_{i=s+1}^{t} a_i \underbrace{E((W_{t_i} - W_{t_{i-1}})|\mathcal{F}_u)}_{=0 \text{ reason see } 3)} \\
&= 0
\end{aligned}
$$

3) $(W_{t_i} - W_{t_{i-1}}) \in [u, t]$ therefore independent of $\mathcal{F}_u$ ☐

Proof 2: We need to show, for $w < v < u < t$, that $X_t - X_u$ and $X_v - X_w$ are independent.
$X_t - X_u = \int_0^t f(s)\,dW_s - \int_0^u f(s)\,dW_s = \int_u^t f(s)\,dW_s = \int \mathbf{1}_{(u,t]} f(s)\,dW_s$
$X_v - X_w = \int \mathbf{1}_{(w,v]} f(s)\,dW_s$

$$
\begin{aligned}
Cov((X_t - X_u), (X_v - X_w)) &= \int \mathbf{1}_{(u,t]} f(s)\,dW_s * \int \mathbf{1}_{(w,v]} f(s)\,dW_s \\
&= \int \underbrace{\mathbf{1}_{(u,t]} * \mathbf{1}_{(w,v]}}_{4)} f^2(s)\,dW_s \\
&= 0
\end{aligned}
$$

Since the Wiener Integral is normally distributed, we can conclude that the covariance of zero implies independence of the increments.

4) $(s, t]$ and $(u, v]$ are disjoint intervals ☐

Group A, B, C Exercise 3

State and prove (sketching the basic idea) the integration by parts rule for Stieltjes integrals.

Proof:
Let $f(s)$ and $g(s)$ be continous FV-functions. Then the integration-by-parts rule says that

$$
f(t)g(t) - f(s)g(s) = \int_s^t f(u)\,dg(u) + \int_s^t g(u)\,df(u)
$$

If we consider a Riemannian sequence of subdivision $[s, t] s.t. 0 = t_0 < t_1 < .. < t_n = t$ then

$$
\begin{aligned}
f(t)g(t) - f(s)g(s) &= \sum_{i=1}^{n} (f(t_i)g(t_i) - f(t_{i-1})g(t_{i-1})) \\
&= \sum_{i=1}^{n} [(f(t_{i-1})(g(t_i) - g(t_{i-1})) + g(t_{i-1})(f(t_i) - f(t_{i-1})) + (g(t_i) - g(t_{i-1}))(f(t_i) - f(t_{i-1}))] \\
&= \underbrace{\sum_{i=1}^{n} (f(t_{i-1})(g(t_i) - g(t_{i-1}))}_{\to \int_s^t f(u)\,dg(u)} + \underbrace{\sum_{i=1}^{n} g(t_{i-1})(f(t_i) - f(t_{i-1}))}_{\to \int_s^t g(u)\,df(u)} \\
&\quad + \underbrace{\sum_{i=1}^{n} (g(t_i) - g(t_{i-1}))(f(t_i) - f(t_{i-1}))}_{\to 0 \text{ reason see 5)}} \\
&= \int_s^t f(u)\,dg(u) + \int_s^t g(u)\,df(u)
\end{aligned}
$$

5) We apply the Cauchy-Schwarz inequality:

$$
0 \leq |\textstyle\sum_{i=1}^{n} (g(t_i) - g(t_{i-1}))(f(t_i) - f(t_{i-1}))| \leq \underbrace{\sqrt{\sum_{i=1}^{n} (g(t_i) - g(t_{i-1}))^2}}_{\text{is the quadratic variation of continous}} \underbrace{\sqrt{\sum_{i=1}^{n} (f(t_i) - f(t_{i-1}))^2}}_{\text{function with FV, so it goes to 0}}
$$

Therefore, $|\sum_{i=1}^{n} (g(t_i) - g(t_{i-1}))(f(t_i) - f(t_{i-1}))|$ goes to 0.

□

Continous Time Finance I
Solution to 3rd Minitest

by Yihui Ni

Group A Exercise 1

Find the covariance of two Ito integrals $\int_0^s H\,dW$ *and* $\int_0^t H\,dW$*?*

We assume that $s < t$ and let's denote $X_s = \int_0^s H\,dW$ and $X_t = \int_0^t H\,dW$ then

Proof 1: By **martingale property**[1)] of Ito Integral

$$\begin{aligned}
Cov(\int_0^s H\,dW, \int_0^t H\,dW) &= Cov(X_s, X_t) \\
&= E(X_s * X_t) + \underbrace{E(X_s)}_{=0}\underbrace{E(X_t)}_{=0} \\
&= E(E(X_s * X_t)|\mathcal{F}_s) \\
&= E(E(\int_0^s H\,dW * \int_0^t H\,dW)|\mathcal{F}_s) \\
&= E(\int_0^s H\,dW * \underbrace{E(\int_0^t H\,dW|\mathcal{F}_s)}_{\text{only measurable up to } s^{1)}}) \\
&= E(\int_0^s H\,dW * \int_0^s H\,dW) \\
{}^{2)}\quad &= E(\int_0^s H^2\,du) \\
&= V(\int_0^s H\,dW)
\end{aligned}$$

□

2) Isometric Equality: $E((\int_0^t H_{s-}dW_s)^2) = E(A_t) = E(\int_0^t H_s^2 ds)$

Proof 2: By **martingale property**[1)] of Ito Integral

$$
\begin{aligned}
Cov\Big(\int_0^s H\,dW, \int_0^t H\,dW\Big) &= Cov(X_s, X_t) \\
&= Cov(X_s, X_t + X_s - X_s) \\
&= Cov(X_s, X_t - X_s) + Cov(X_s, X_s) \\
&= E(X_s * (X_t - X_s)) - \underbrace{E(X_s)}_{=0} * E(X_t - X_s) + V(X_s) \\
&= E(E(X_s * (X_t - X_s))|\mathcal{F}_s) + V(X_s) \\
&= E(E([\int_0^s H\,dW * (\int_0^t H\,dW - \int_0^s H\,dW)]|\mathcal{F}_s)) + V(\int_0^s H\,dW) \\
&= E(\int_0^s H\,dW * \underbrace{\quad E(\int_s^t H\,dW|\mathcal{F}_s)\quad}_{=0\text{, only measurable up to } s^{1)}}) + V(\int_0^s H\,dW) \\
&= V(\int_0^s H\,dW)
\end{aligned}
$$

□

Comment: Please note that
$Cov(\int_0^s H\,dW, \int_0^t H\,dW) = V(\int_0^s H\,dW) = E(\int_0^s H^2\,du)$

but

$E(\int_0^s H^2\,du) \neq \int_0^s H^2\,du$ **(-0.5 points)**
$V(\int_0^s H\,dW) \neq \int_0^s H^2\,du$ **(-0.5 points)**

Group A Exercise 2

Check square integrability and find expectation and variance of $\int_0^t e^{W_s+2}\,dW_s$

$$
\begin{aligned}
E(\int_0^t e^{W_s+2}\,dW_s)^2 &\overset{2)}{=} E(\int_0^t e^{2W_s+4}\,ds) \\
&= \int_0^t E(e^{2W_s+4})\,ds \\
&\overset{3)}{=} \int_0^t e^{4+\frac{4s}{2}}\,ds \\
&= \int_0^t e^{4+2s}\,ds \\
&= \frac{1}{2}e^{4+2s}\,|_0^t \\
&= \frac{1}{2}e^4(e^{2t}-1) < \infty
\end{aligned}
$$

Therefore,
$E(\int_0^t e^{W_s+2}\,dW_s) = 0$
$V(\int_0^t e^{W_s+2}\,dW_s) = E(\int_0^t e^{W_s+2}\,dW_s)^2 = \frac{1}{2}e^4(e^{2t}-1)$

2) Isometric Equality: $E((\int_0^t H_{s-}dW_s)^2) = E(A_t) = E(\int_0^t H_s^2 ds)$
3) $E(e^{\sigma Z}) = e^{\frac{\sigma^2}{2}}$ where Z is a normally distributed random veriable and σ is its standard deviation

Group B Exercise 1

Find the covariance of two Ito integrals $\int_0^t H\,dW$ *and* $\int_0^t G\,dW$*?*

$Cov(X,Y) = \frac{V(X+Y)-V(X)-V(Y)}{2}$

$$
\begin{aligned}
Cov(\int_0^t H\,dW, \int_0^t G\,dW) &= \frac{V(\int_0^t H\,dW + \int_0^t G\,dW) - V(\int_0^t H\,dW) - V(\int_0^t G\,dW)}{2} \\
&= \frac{E(\int_0^t (H+G)\,dW)^2 - E(\int_0^t H\,dW)^2 - E(\int_0^t G\,dW)^2}{2} \\
&\overset{2)}{=} \frac{E(\int_0^t (H+G)^2\,ds) - E(\int_0^t H^2\,ds) - E(\int_0^t G^2\,ds)}{2} \\
&= \frac{E(\int_0^t H^2\,ds) + E(\int_0^t G^2\,ds) - E(\int_0^t H^2\,ds) - E(\int_0^t G^2\,ds)}{2} \\
&+ \frac{2 * E(\int_0^t (H * G)\,ds)}{2} \\
&= E(\int_0^t (H * G)\,ds)
\end{aligned}
$$

2) Isometric Equality: $E((\int_0^t H_{s-} dW_s)^2) = E(A_t) = E(\int_0^t H_s^2 ds)$

Group B Exercise 2

Check square integrability and find expectation and variance of $\int_0^t e^{2-W_s}\,dW_s$

$$
\begin{aligned}
E(\int_0^t e^{2-W_s}\,dW_s)^2 &\overset{2)}{=} E(\int_0^t e^{4-2W_s}\,ds) \\
&= \int_0^t E(e^{4-2W_s})\,ds \\
&\overset{3)}{=} \int_0^t e^{4+\frac{4s}{2}}\,ds \\
&= \int_0^t e^{4+2s}\,ds \\
&= \frac{1}{2} e^{4+2s} \,|_0^t \\
&= \frac{1}{2} e^4 (e^{2t} - 1) < \infty
\end{aligned}
$$

Therefore,
$E(\int_0^t e^{2-W_s}\,dW_s) = 0$
$V(\int_0^t e^{2-W_s}\,dW_s) = E(\int_0^t e^{2-W_s}\,dW_s)^2 = \frac{1}{2} e^4 (e^{2t} - 1)$

2) Isometric Equality: $E((\int_0^t H_{s-} dW_s)^2) = E(A_t) = E(\int_0^t H_s^2 ds)$

3) $E(e^{\sigma Z}) = e^{\frac{\sigma^2}{2}}$ where Z is a normally distributed random veriable and σ is its standard deviation

Group C Exercise 1

Show that Ito Integral have uncorrelated increments?

We assume that $s < t$ and let's denote $X_0 = 0,\ X_s = \int_0^s H\, dW,\ X_t = \int_0^t H\, dW$ and therefore $X_t - X_s = \int_s^t H\, dW$

Proof 1: By Martingale Property[1)] of Ito Integral

$$\begin{aligned}
Cov(\int_0^s H\, dW, \int_s^t H\, dW) &= Cov((X_s - X_0), (X_t - X_s)) \\
&= Cov(X_s, (X_t - X_s)) \\
&= E(X_s * (X_t - X_s)) - \underbrace{E(X_s)}_{=0} * E(X_t - X_s) \\
&= E(E(X_s * (X_t - X_s))|\mathcal{F}_s) \\
&= E(X_s * E((X_t - X_s)|\mathcal{F}_s)) \\
&= E(X_s * [\underbrace{E(X_t|\mathcal{F}_s)}_{=X_s 1)} - X_s]) \\
&= E(X_s * 0) \\
&= 0
\end{aligned}$$

□

Proof 2: By Martingale Property[1)] of Ito Integral

$$\begin{aligned}
Cov(\int_0^s H\, dW, \int_s^t H\, dW) &= Cov((X_s - X_0), (X_t - X_s)) \\
&= Cov(X_s, (X_t - X_s)) \\
&= Cov(X_s, X_t) - Cov(X_s, X_s) \\
&= Cov(X_s, X_t) - Cov(X_s, X_s) \\
&= E(X_s * X_t) - \underbrace{E(X_s)}_{=0} \underbrace{E(X_t)}_{=0} - V(X_s) \\
&= E(E(X_s * X_t)|\mathcal{F}_s)) - V(X_s) \\
&= E(X_s * \underbrace{E(X_t|\mathcal{F}_s)}_{=X_s 1)}) - E(X_s^2) \\
&= 0
\end{aligned}$$

□

Proof 3: Let $s < t < u < v$

$$
\begin{aligned}
Cov(\int_0^s H\,dW, \int_s^t H\,dW) &= Cov(\int \mathbb{1}_{[s,t)} H\,dW, \int \mathbb{1}_{[u,v)} H\,dW) \\
&= E(\int \mathbb{1}_{[s,t)} H\,dW * \int \mathbb{1}_{[u,v)} H\,dW) \\
{}^{2)} &= E(\int \mathbb{1}_{[s,t)} * \mathbb{1}_{[u,v)}\, H^2\,dx) \\
&= 0
\end{aligned}
$$

□

2) Isometric Equality: $E((\int_0^t H_{s-} dW_s)^2) = E(A_t) = E(\int_0^t H_s^2 ds)$

Group C Exercise 2

Check square integrability and find expectation and variance of $\int_0^t W_s^2\,dW_s$

$$
\begin{aligned}
E(\int_0^t W_s^2\,dW_s)^2 &\overset{2)}{=} E(\int_0^t W_s^4\,ds) \\
&= \int_0^t E(W_s^4)\,ds \\
&\overset{4)}{=} \int_0^t 3s^2\,ds \\
&= x^3\mid_0^t \\
&= t^3 < \infty
\end{aligned}
$$

Therefore,
$E(\int_0^t W_s^2\,dW_s) = 0$
$V(\int_0^t W_s^2\,dW_s) = E(\int_0^t W_s^2\,dW_s)^2 = t^3$

2) Isometric Equality: $E((\int_0^t H_{s-} dW_s)^2) = E(A_t) = E(\int_0^t H_s^2 ds)$
4) $E((\sigma X)^4) = \sigma^4 E(X^4) = 3\sigma^4 = 3 * V(X)^2$ where X is a normally distributed random veriable, $V(X)$ is its variance and σ is its standard deviation

Group A, B, C Exercise 3

State the basic assertion about the martingale property of an Ito integral and of its square.

Theorem 3.5
Let $X_t = \int_0^t H_s \, dW_s$
If $E(A_t) = E(\int_0^t H_s^2 \, ds) < \infty$ (therefore A_t is integrable), then

1. (X_t) is a square integrable martingale with $(E(X_t) = 0, V(X_t) = E(\int_0^t H_s^2 \, ds))$ and continous paths **(1 point)**
2. $(X_t^2 - A_t)$ is a martingale **(1 point)**

   Also good to know:

3. (X_t) is well-defined for any adapted cadlag process (H_s)
4. (X_t) is linear in the integrand and the integrator, and satisfies the jump rule, the associativity rule and the stopping rule

Continous Time Finance I
Solution to 4th Minitest

by Yihui Ni

Group A Exercise 1

Expand by integration by parts $(2t-3W_t)^2$.

Version 1:
$(2t-3W_t)^2 = 4t^2 - 12tW_t + 9W_t^2$

$dt^2 = tdt + tdt = 2tdt$
$d(tW_t) = tdW_t + W_tdt + \underbrace{d[t,W]_t}_{=0} = tdW_t + W_tdt$
$dW_t^2 = W_tdW_t + W_tdW_t + d[W,W]_t = 2W_tdW_t + dt$

$$\begin{aligned} d(2t-3W_t)^2 &= 8tdt - 12tdW_t - 12W_tdt + 18W_tdW_t + 9dt \\ &= (8t - 12W_t + 9)dt + (18W_t - 12t)dW_t \end{aligned}$$

Version 2:
$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X,Y]_t$

$$\begin{aligned}(2t-3W_t)^2 &= (2t-3W_t)(2t-3W_t)\\ &= \underbrace{(2*0-3W_0)(2*0-3W_0)}_{=0}\\ &+ \int_0^t (2s-3W_s)d(2s-3W_s) + \int_0^t (2s-3W_s)d(2s-3W_s)\\ &+ [\underbrace{2s}_{1)} -3W_s, \underbrace{2s}_{1)} -3W_s]_t\end{aligned}$$

$$\begin{aligned}&= 2(\int_0^t (2s-3W_s)d(2s) - \int_0^t (2s-3W_s)d(3W_s))\\ &+ [-3W_s, -3W_s]_t\end{aligned}$$

$$\begin{aligned}&= 2(\int_0^t 4sds - \int_0^t 6W_sds - \int_0^t 6sdW_s + \int_0^t 9W_sdW_s)\\ &+ 9\underbrace{[W_s, W_s]_t}_{=9t\ 2)}\end{aligned}$$

$$= 2(\int_0^t (4s-6W_s)ds - \int_0^t (6s-9W_s)dW_s) + 9t$$

1)this does not contribute to quadratic variation
2)quadratic variation of Wiener process

Version 3:
$(2t-3W_t)^2 = 4t^2 - 12tW_t + 9W_t^2$

$tW_t = \int_0^t tdW_t + \int_0^t W_tdt + \underbrace{d[t,W]_t}_{=0} = tdW_t + W_tdt$

$W_t^2 = \int_0^t 2W_tdW_t + t$

$$\begin{aligned}(2t-3W_t)^2 &= 4t^2 - 12\int_0^t sdW_s - 12\int_0^t W_sds + 18\int_0^t W_sdW_s + 9t\\ &= 4t^2 + 9t - \int_0^t (12t-18W_s)dW_s - 12\int_0^t W_sds\end{aligned}$$

Group A Exercise 2

Represent $f(W_t)$ as an Ito-process when $f(x) = 2x^3 - 1$

Version 1: $f(W_t) = 2W_t^3 - 1$
$f(W_t) = f(W_0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds$

$f'(x) = 6x^2$
$f''(x) = 12x$

$$\begin{aligned} f(W_t) &= 2W_t^3 - 1 \\ &= -1 + \int_0^t 6W_s^2 dW_s + \frac{1}{2}\int_0^t 12W_s ds \end{aligned}$$

Please note that for the solution with integral, one must not forget **-1**! **(-0.5 Points)**
For the shorthand version, **-1** has to be left out. **(-0.5 Points)**

shorthand version:

$$\begin{aligned} df(W_t) &= d(2W_t^3 - 1) \\ &= 6W_t^2 dW_t + 6W_t dt \end{aligned}$$

Group A Exercise 3

Find the quadratic variation of $\int_0^t W_s dW_s^2$.

$\int_0^t W_s dW_s^2 = \int_0^t W_s d[2W_s dW_s + ds] = 2\int_0^t W_s^2 dW_s + \underbrace{\int_0^t W_s ds}_{3)}$

$[\int W_s dW_s^2]_t = [2\int W^2 dW]_t = 4\int_0^t W^4 d\underbrace{[W,W]_s}_{=t\ 2)} = 4\int_0^t W_s^4 ds$

2) quadratic variation of Wiener process
3)continues and FV therefore quadratic variation would be 0

Group B Exercise 1

Expand by integration by parts $(W_t+1)^2$.

Version 1:
$(W_t+1)^2 = W_t^2 + 2W_t + 1$

$$\begin{aligned} d(W_t+1)^2 &= dW_t^2 + 2dW_t \\ &= 2W_t dW_t + dt + 2dW_t \\ &= (2W_t+2)dW_t + dt \end{aligned}$$

Version 2:

$$\begin{aligned} (W_t+1)^2 &= (W_0+1)(W_0+1) + 2\int_0^t (W_s+1)d(W_s+1) + [(W_s+1),(W_s+1)]_t \\ &= 1 + 2\int_0^t (W_s+1)dW_s + [W,W]_t \\ &= 1 + 2\int_0^t (W_s+1)dW_s + t \\ &= 1 + 2\int_0^t W_s dW_s + 2\underbrace{\int_0^t 1dW_s}_{7)} + t \\ &= 1 + 2\int_0^t W_s dW_s + 2W_t + t \end{aligned}$$

7) $\int_0^t 1d(W_s+1) = W_t$

Group B Exercise 2

Represent $f(W_t)$ *as an Ito-process when* $f(x) = e^{1-x}$

$f(W_t) = e^{1-W_t}$

$f(W_t) = f(W_0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds$

$f'(x) = -e^{1-x}$
$f''(x) = e^{1-x}$

$$\begin{aligned} df(W_t) &= de^{1-W_t} \\ &= -e^{1-W_t}dW_t + \frac{1}{2}e^{1-W_t}dt \end{aligned}$$

Version 2:

$$
\begin{aligned}
f(W_t) &= e^{1-W_0} + (-1)\int_0^t e^{1-W_s} dW_s + \frac{1}{2}\int_0^t e^{1-W_s} ds \\
&= e - \int_0^t e^{1-W_s} dW_s + \frac{1}{2}\int_0^t e^{1-W_s} ds
\end{aligned}
$$

Please note that for the solution with integral, one must not forget **e**!
(-0.5 Points)
For the shorthand version, **e** has to be left out.

Group B Exercise 3

Find the quadratic variation of W_t^3.

(Example 4.8 ii): integration by part formula: $dW_t^3 = 3W_t^2 dW_t + 3W_t dt$

$[W_t^3] = [3\int W_t^2 dW_t]_t = 9\int_0^t W_s^4 d[W,W]_s = 9\int_0^t W_s^4 ds$

Group C Exercise 1

Expand by integration by parts tW_t^2.

$$
\begin{aligned}
d(tW_t^2) &= t dW_t^2 + W_t^2 dt + \underbrace{[W,t]_t}_{=0} \\
&= t d(2W_t dW_t + dt) + W_t^2 dt \\
&= (t + W_t^2)dt + 2tW_t dW_t
\end{aligned}
$$

Version 2:
$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X,Y]_t$

$$
\begin{aligned}
\mathrm{t}W_t^2 &= 0 * W_0 + \int_0^t s dW_s^2 + \int_0^t W_s^2 ds + \underbrace{[s, W_s^2]_t}_{=0 \; 5)} \\
&= \int_0^t s dW_s^2 + \int_0^t W_s^2 ds
\end{aligned}
$$

$W_s^2 = 2\int_0^t W_s dW_s + t$

$dW_s^2 = 2W_s dW_s + dt$

$$\begin{aligned}
\text{tW}_t^2 &= \int_0^t s dW_s^2 + \int_0^t W_s^2 ds \\
&= \int_0^t s(2W_s dW_s + ds) + \int_0^t W_s^2 ds \\
&= \int_0^t s2W_s dW_s + \int_0^t s ds + \int_0^t W_s^2 ds \\
&= \int_0^t 2sW_s dW_s + \int_0^t (s + W_s^2) ds
\end{aligned}$$

5) Continous, FV therefore quadratic variation is zero

Group C Exercise 2

Represent $f(W_t)$ as an Ito-process when $f(x) = e^{-x}$

$f(W_t) = e^{-W_t}$

$f(W_t) = f(W_0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)ds$

$f'(x) = -e^{-x}$
$f''(x) = e^{-x}$

$$\begin{aligned}
df(W_t) &= de^{-W_t} \\
&= -e^{-W_t}dW_t + \frac{1}{2}e^{-W_t}dt
\end{aligned}$$

Version 2:

$$\begin{aligned}
f(W_t) &= e^{-W_0} + (-1)\int_0^t e^{-W_s}dW_s + \frac{1}{2}\int_0^t e^{-W_s}ds \\
&= 1 - \int_0^t e^{-W_s}dW_s + \frac{1}{2}\int_0^t e^{-W_s}ds
\end{aligned}$$

Please note that for the solution with integral, one must not forget **1**!
(-0.5 Points)
For the shorthand version, **1** has to be left out.

Group C Exercise 3

Find the quadratic variation of $W_t^2 - W_t$.

Version 1: $W_t^2 - W_t = 2\int_0^t W dW + t - W_t = \int_0^t (2W_s - 1)dW_s + t$

$[W^2 - W]_t = \int_0^t (2W_s - 1)^2 ds$

Version 2:

$$\begin{aligned}
[W^2 - W]_t &= [W^2 - W, W^2 - W]_t \\
&= [W^2, W^2]_t - 2[W^2, W]_t + \underbrace{[W, W]_t}_{=t\ 2)} \\
&= [2\int_0^t W_s dW_s + \underbrace{t}_{1)}]_t - 2[2\int_0^t W_s dW_s + \underbrace{t}_{1)}, W]_t + \underbrace{[W, W]_t}_{=t} \\
&= [2\int_0^t W_s dW_s]_t - 2[2\int_0^t W_s dW_s, W]_t + t \\
&= 4\int_0^t W_s^2 ds - 4\int_0^t W_s ds + t
\end{aligned}$$

1)this does not contribute to quadratic variation
2)quadratic variation of Wiener process

Please not that e.g.
$[2\int_0^t W_s dW_s + W_s] \neq 4\int_0^t W_s^2 ds + W_s^2$ **(-1.5 Points)**
$[2\int_0^t W_s dW_s + W_s] \neq 4\int_0^t W_s^2 ds$ **(-1.5 Points)**
Solution see before

Continous Time Finance I
Solution to 5th Minitest

by Yihui Ni

Group A Exercise 1

Let $X_t = \int_0^t W\, dW$. *Represent as an Ito-process:* X_t^3

$X_t = \int_0^t W_s dW_s \qquad X_0 = 0$
$X_t^3 = 0 + \int_0^t 3X_s^2\, dX_s + \frac{1}{2}\int_0^t 6X_s d[X]_s$

$dX_t = W_t dW_t \qquad [X]_t = \int_0^t W_s^2 ds \qquad d[X]_t = W_t^2 dt$

$X_t^3 = \int_0^t 3X_s^2 W_s dW_s + \int_0^t 3X_s W_s^2 ds$

Not necessary to replace $X_t = \frac{1}{2}(W_t^2 - t)$

Group A Exercise 2

Solve the stochastic differential equiation $dS_t = -S_t dt + dW_t$

$dS_t = -S_t dt + dW_t$
$dS_t + S_t dt = dW_t$

Multiplication by e^t
$e^t dS_t + S_t e^t dt = e^t dW_t$
$e^t dS_t + S_t de^t = e^t dW_t$

$d(e^t S_t) = e^t dW_t$

$e^t S_0 - e^0 S_0 = \int_0^t e^s dW_s$

$S_t = e^{-t} S_0 + \int_0^t e^{s-t} dW_s$

Group B Exercise 1

Let $X_t = \int_0^t W\, dW$. *Represent as an Ito-process:* $e^{X_t^2}$

$\rho(0) = e^0 = 1$

$\rho(x) = e^{x^2}$
$\rho'(x) = 2xe^{x^2}$
$\rho''(x) = 2e^{x^2}$+4x$^2 ex^2$

Version 1:
$e^{X_t^2} = 1 + \int_0^t 2X_s e^{X_s^2} dX_s + \frac{1}{2}\int_0^t (2e^{X_s^2} + 4X_s^2 e^{X_s^2}) d[X]_s$

$dX_t = W_t dW_t \qquad d[X]_t = W_t^2 dt$

$e^{X_t^2} = 1 + \int_0^t 2X_s e^{X_s^2} W_s dW_s + \int_0^t (e^{X_s^2} + 2X_s^2 e^{X_s^2}) W_s^2 ds$

Version 2:
$d\,(e^{X_t^2}) = 2X_t e^{X_t^2} dX_t + (2e^{X_t^2} + 4X_t^2 e^{X_t^2}) d[X]_t$

$dX_t = W_t dW_t \qquad d[X]_t = W_t^2 dt$

$d\,(e^{X_t^2}) = 2X_t e^{X_t^2} W_t dW_t + (e^{X_t^2} + 2X_t^2 e^{X_t^2}) W_t^2 dt$

Please note for *Version 1*: $e^0 = \mathbf{1}$ part is needed.
Please note for *Version 2*: $e^0 = \mathbf{1}$ part must not be.

Group B Exercise 2

Solve the stochastic differential equiation $dS_t = -S_t dt + S_t dW_t$
$dS_t = -S_t dt + S_t dW_t = S_t(-dt + dW_t) = S_t d(W_t - t)$

$S_t = S_0 \sum (W_t - t)$

$= S_0 eW_t - t - \frac{1}{2}[W - t]_t$

$= S_0 eW_t - t - \frac{t}{2}$

$=_0 eW_t - \frac{3t}{2}$

NOT CORRECT:
$dS_t = -S_t dt + S_t dW_t$
$dS_t + S_t dt = S_t dW_t$

Multiplication by e^t

$e^t dS_t + S_t e^t dt = e^t S_t dW_t$
$d(e^t S_t) = e^t S_t dW_t$
$\not\Leftrightarrow e^t S_t = S_0 + \int_0^t e^s S_s dW_s$

It is not correct since there would be 2 terms depending on t (e^t and S_t).
Wieder andere behaupten:
$\varepsilon(x)_t = e^{X_t - \frac{[X]_t}{2}}$

$S_t = S_0 \varepsilon(W_t^2) = S_0 e^{W_t - \frac{[W]_t}{2}} = S_0 e^{W_t - 2\int_0^t W_s^2 ds}$

$W_t^2 = 2\int_0^t W_s dW_s + t$

$[W^2]_t = [2\int_0^t W_s dW_s + t]_t = [2\int_0^t W_s dW_s + t, 2\int_0^t W_s dW_s + t]_t = 4\int_0^t W_s^2 ds$

Others uses the same method but claim but does not make sense at all:

$[dW^2]_t = t$

Group C Exercise 1

Let $X_t = \int_0^t W\, dW$. *Represent as an Ito-process:* $e^{\mu t + \sigma X_t}$

$\rho(0, x_0) = e^0 = 1$

$\rho(t, x) = e^{\mu t + \sigma x}$
$\rho_t(t, x) = \mu e^{\mu t + \sigma x}$
$\rho_x(t, x) = \sigma e^{\mu t + \sigma x}$
$\rho_{xx}(t, x) = \sigma^2 e^{\mu t + \sigma x}$

Version 1:
$e^{\mu s + \sigma X_s} = 1 + \int_0^t \mu e^{\mu s + \sigma x}\, ds + \int_0^t \sigma e^{\mu s + \sigma x}\, dX_s + \frac{1}{2}\int_0^t \sigma^2 e^{\mu s + \sigma x} d[X]_s$

$dX_t = W_t dW_t \qquad d[X]_t = W_t^2 dt \qquad$ and let $Y_t = e^{\mu t + \sigma x}$

$e^{\mu s + \sigma X_s} = 1 + \int_0^t \mu Y_s\, ds + \int_0^t \sigma Y_s\, W_s dW_s + \frac{1}{2}\int_0^t \sigma^2 Y_s W_s^2 ds$

Version 2:
$e^{\mu t + \sigma X_t} = \mu e^{\mu t + \sigma x}\, dt + \sigma e^{\mu t + \sigma x}\, dX_t + \frac{1}{2}\sigma^2 e^{\mu t + \sigma x} d[X]_t$

$dX_t = W_t dW_t \qquad d[X]_t = W_t^2 dt \qquad$ and let $Y_t = e^{\mu t + \sigma x}$

$e^{\mu t + \sigma X_t} = \mu Y_t\, dt + \sigma Y_t\, W_t dW_t + \frac{1}{2}\sigma^2 Y_t W_t^2 dt$

Please note for *Version 1*: $e^0 = \mathbf{1}$ part is needed.
Please note for *Version 2*: $e^0 = \mathbf{1}$ part must not be.
Not necessary to replace $X_t = \frac{1}{2}(W_t^2 - t)$

Group C Exercise 2

Solve the stochastic differential equiation $dS_t = -S_t dt + S_t dW_t$
$dS_t = S_0 \varepsilon(W_t^2) = S_0 e^{W_t^2 - \frac{[W^2]_t}{2}}$

$W_t^2 = 2\int_0^t W dW + t$

$[W^2]_t = 4\int_0^t W_s^2 ds$

$S_t = S_0 e^{2\int_0^t W dW + t - 2\int_0^t W_s^2 ds}$

$S_t = S_0 e^{2\int_0^t W dW + {}^t(1-2W_s^2)ds}$

Group C Exercise 3

Let X_t *be a continous semimartingale. Prove that* $S_t = S_0 e^{X_t - [X]_t/2}$

$Abbreviation: Y_t = X_t - \frac{1}{2}[X]_t$

$S_t = S_0 e^{Y_t}$

Ito -Formula:
$dS_t = S_0 de^{Y_t} = S_0(e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} d[Y]_t)$

Comment: some students claims that $d[Y]_t$ is a FV process and therefore $\frac{[X_t]}{2} = 0$ I believe this is not correct

$\mathbf{Y}_t = X_t - \frac{1}{2}[X]_t$

$[Y]_t = [X]_t \qquad d[Y]_t = d[X]_t$

$dS_t = S_0(e^{Y_t} dX_t - \frac{1}{2} e^{Y_t} d[X]_t + \frac{1}{2} e^{Y_t} d[X]_t)$

$dS_t = S_0 e^{Y_t} dX_t = S_t dX_t$